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Character expansion for HOMFLY polynomials. I. Integrability and difference equations

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ABSTRACT

We suggest to associate with each knot the set of coefficients of its HOMFLY polynomial expansion into the Schur functions. For each braid representation of the knot these coefficients are defined unambiguously as certain combinations of the Racah symbols for the algebra SU_q. Then, the HOMFLY polynomials can be extended to the entire space of time-variables. The so extended HOMFLY polynomials are no longer knot invariants, they depend on the choice of the braid representation, but instead one can naturally discuss their explicit integrable properties. The generating functions of torus knot/link coefficients are turned to satisfy the Plücker relations and can be associated with τ-function of the KP hierarchy, while generic knots correspond to more involved systems. On the other hand, using the expansion into the Schur functions, one can immediately derive difference equations (A-polynomials) for knot polynomials which play a role of the string equation. This adds to the previously demonstrated use of these character decompositions for the study of β-deformations from HOMFLY to superpolynomials.

To the memory of Max Kreuzer

1 Introduction

Knot theory is a very old and complicated mathematical domain, with many deep ideas and results. Its counterpart in quantum field theory is the $3d$ Chern-Simons (CS) model [1] and its extensions to higher dimensions. For string/M-theory of main interest are various Wilson averages in CS theory and, most important, relations between them. These Wilson averages are known in CS theory as HOMFLY "polynomials" [2], while some of the relations (some linear ones) have appeared under the name of "quantum A-polynomial" [3]. Knot theory is becoming especially interesting today, because there is now a strong belief that the HOMFLY polynomials are closely related to KP/Toda τ-functions, providing a minor deformation of these, while the linear relations provide likewise minor deformations of the string equations and the Virasoro constraints [4]. Since by now a lot is known on the knot *phenomenology*, e.g. concrete HOMFLY polynomials are easily available, say, at [5, 6], the time is

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coming to proceed to a *theoretical* analysis of the *problem*, which can be, *in particular*, nicknamed as

$$\boxed{\textbf{HOMFLY polynomials as deformed matrix model } \tau - \textbf{functions}}\ ^1$$

There are already numerous attempts in the literature, targeted at this problem (perhaps, not formulated so explicitly), see, for example, [9].

Our suggestion to attack the problem, after it is explicitly formulated, is to rely upon the known common property of the polynomial KP τ-functions and HOMFLY polynomials: **these both can be expanded into the Schur functions**, the characters of the linear group $GL(\infty)$. The Schur functions $S_Q\{p\}$ depend on an infinite set of "time-variables" $p_k = kt_k$, $k = 1, 2, \dots$ (p and t are the two standard choices, widely used in different fields), and correspond to the representations of the linear group or, simply, are labeled by the Young diagrams $Q = \{\lambda_1 \geq \lambda_2 \geq \dots \geq 0\}$.

The KP τ-functions (solutions to the bilinear Hirota equation) are linear combinations of the Schur functions (see [4, 10] for reviews in the terms relevant for our purposes):

$$\tau\{p\,|g\} = \sum_Q g_Q S_Q\{p\}, \tag{1}$$

provided the coefficients g_Q satisfy the infinite set of bilinear Plücker relations,

$$\begin{gathered}
g_{[22]}g_{[0]} - g_{[21]}g_{[1]} + g_{[2]}g_{[11]} = 0, \\
\hline
g_{[32]}g_{[0]} - g_{[31]}g_{[1]} + g_{[3]}g_{[11]} = 0, \\
g_{[221]}g_{[0]} - g_{[211]}g_{[1]} + g_{[2]}g_{[111]} = 0, \\
\hline
g_{[42]}g_{[0]} - g_{[41]}g_{[1]} + g_{[4]}g_{[11]} = 0, \\
g_{[33]}g_{[0]} - g_{[31]}g_{[2]} + g_{[3]}g_{[21]} = 0, \\
g_{[321]}g_{[0]} - g_{[311]}g_{[1]} + g_{[3]}g_{[111]} = 0, \\
g_{[222]}g_{[0]} - g_{[211]}g_{[11]} + g_{[21]}g_{[111]} = 0, \\
g_{[2211]}g_{[0]} - g_{[2111]}g_{[1]} + g_{[2]}g_{[1111]} = 0, \\
\dots
\end{gathered} \tag{2}$$

and the (infinite) set of coefficients $g = \{g_Q\}$ describes a point in the Universal Grassmannian [11]. Moreover, one further generalizes τ in (1) to be a Toda-lattice τ-function, provided the coefficients g_R themselves depend on another infinite set of time variables, $\bar{p}_k$, and

$$g_Q = \sum_R g_Q^R S_R\{\bar{p}\} \tag{3}$$

where g_Q^R satisfy some more involved bilinear relations. For matrix model like τ-functions, which *actually* arise in the role of generating functions in quantum field theory, these coefficients also satisfy *linear* relations, known as string equations or, more generally, Virasoro constraints.

[1] We remind that the matrix model τ-functions is a particular class of τ-functions which, in addition to the bilinear Hirota equations, also satisfy a linear "string equation" generating together with the Hirota equations an entire set of Virasoro like constraints. These constraints can be also described in terms of the AMM/EO topological recursion [7]. Sometimes (e.g. in β-ensembles) only these Virasoro like constraints are known, while bilinear equations (and associated Harer-Zagier recursion) remain to be found [8]. It seems that in knot theory we also typically know only linear equations (A-polynomial). In this paper we find the bilinear identities, at least, in the case of torus knots.

The HOMFLY polynomial is equal to the properly defined[2] Wilson loop average in CS theory with the group $SU(N)$ and the coupling constant $q = \exp\left(\frac{2\pi i}{k+N}\right)$:

$$H_R^{\mathcal{K}} = \left\langle \operatorname{tr}_R P \exp \oint_{\mathcal{K}} \mathcal{A} \right\rangle_{CS(N,q)} \tag{4}$$

Usually the N-dependence is traded for A-dependence, where $A = q^N$. Then H_R is a polynomial in A (modulo some common power of A that depends on the normalization). It is labeled by the representation index R and already in this respect resembles the coefficients g_R in (1), only the role of point of the Universal Grassmannian is now played by the triple $(\mathcal{K}, A, q)$. However, for $q \neq 1$ these $H_R^{\mathcal{K}}$ do *not* satisfy the Plücker relations (2),[3] thus the generating function

$$\mathfrak{H}\{p|\mathcal{K}\} = \sum_R H_R^{\mathcal{K}} S_R\{p\} \tag{5}$$

is some q-deformation of the KP τ-function, which still remains to be investigated and understood.

However, the HOMFLY polynomials themselves possess another expansion, similar to (3):

$$\boxed{H_R^{\mathcal{K}} = \sum_Q h_R^Q S_Q\{p^*\} \equiv \mathcal{H}_R\{p^*|\mathcal{K}\}} \tag{6}$$

and relations (linear and non-linear) between the $\mathcal{K}$-dependent coefficients h_R^Q are the ones to be found. Like (1) and like (a very different) Vassiliev-Kontsevich expansion in knot theory, (6) separates dependencies on different variables, in this case on the group (which is contained in the time-variables) and on the knot, which are strongly mixed in the original definitions (either through CS theory or directly through braid representations, for an overview of their still obscure connection see [12]). An important difference from (1) and (3) is that p^* in (6) is not an *arbitrary* point in the space of time-variables: it is constrained to just a 2-dimensional slice

$$p_k^* = \frac{A^k - A^{-k}}{q - q^{-1}} = \frac{\{A^k\}}{\{q\}} \tag{7}$$

Hereafter, we introduced a useful notation $\{x\} = x - x^{-1}$ to simplify the formulas. For $A = q^N$ these $p_k^* = [N]_q \equiv \{q^N\}/\{q\}$.

The manifest expressions for the Schur functions $S_Q\{p^*\}$ in these special points (7) are quite simple and generalize the standard hook formula [16]:

$$S_Q\{p^*\} = \prod_{(i,j)\in Q} \frac{\{Aq^{i-j}\}}{\{q^{h_{i,j}}\}} \quad \stackrel{A=q^N}{\longrightarrow} \quad \prod_{(i,j)\in Q} \frac{[N+i-j]_q}{[h_{i,j}]_q} \tag{8}$$

where $h_{i,j}$ is the hook length.

Given (6), one can easily continue $\mathcal{H}_R\{p|\mathcal{K}\}$ to arbitrary values of p, where it can be compared with KP/Toda τ-functions. The problem is, however, to define the coefficients h_R^Q. For most knots

[2] See [12, 13, 14] for recent review of *existing* problems.

[3] For instance, consider the particular case of the HOMFLY polynomial at $N = 2$, i.e. $A = q^2$. Then, for the spin j representation this knot polynomial is the Jones polynomial J_{2j+1} and the simplest Plücker relation in (2) looks like

$$g_{[0]} = J_1(\mathcal{K}) = 1, \qquad g_{[1]} = J_2(\mathcal{K}), \qquad g_{[2]} = J_3(\mathcal{K}), \qquad g_{[11]} = J_1(\mathcal{K}) = 1, \qquad g_{[21]} = J_2(\mathcal{K}), \qquad g_{[22]} = J_1(\mathcal{K}) = 1,$$

$$g_{[22]}g_{[0]} - g_{[21]}g_{[1]} + g_{[2]}g_{[11]} = 1 - J_2^2(\mathcal{K}) + J_3(\mathcal{K})$$

At the same time, from the relation $J_{R^{\otimes m}}(\mathcal{K}) = J_R(\mathcal{K}^m)$, where $\mathcal{K}^m$ denotes m-cabling of the knot $\mathcal{K}$ (see, e.g., [15, eq.(1.5b)]) it follows that $1 + J_3(\mathcal{K}) = J_2(\mathcal{K}^2) \neq \Big(J_2(\mathcal{K})\Big)^2$ (unless $\mathcal{K}$ is unknot). Therefore, already the first relation in (2) is not fulfilled.

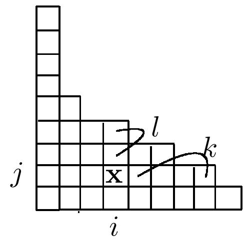


Figure 1: The figure which illustrates the notation in the generalization of the standard hook formula to the quantum dimensions (8). Here the cross $\mathbf{x}$ corresponds to the box of the Young diagram with coordinates (i,j). The corresponding hook length is equal to $h_{i,j} = k + l + 1$.

(represented by braids with more than 3 strands) they can not be obtained from the known expressions for HOMFLY polynomials: for $|Q| \geq 4$ the $S_Q^* = S_Q\{p^*\}$ form a linearly dependent set of functions of the A-variable (while q-dependence does not help, since h_Q can also depend on q). In [17] we suggested to overcome this problem by considering the HOMFLY polynomials for series of knots at once and with the β-deformation [18] switched on (i.e. the "superpolynomials") [19]: then the decomposition like (6) becomes unambiguous and all the coefficients h_R^Q can be found. This is a very promising and interesting direction. However, there is an alternative approach directly based on (quantum) group theory underlying the CS theory and using the Reshetikhin-Turaev construction for the HOMFLY polynomials [20] which arises in the temporal gauge, $A_0 = 0$ and which we basically exploit here. The details of the approach can be found in a separate paper [6], here we only briefly describe the scheme in section 2.

The remaining two sections describe two immediate applications of the character expansion of the HOMFLY polynomials. In section 3 we discuss integrable properties of various knots and explain that in the case of torus knots the generating function of $\mathcal{H}_R\{\bar{p}|\mathcal{K}\}$ is a KP τ-function in variables $\bar{t}_k = p_k/k$:

$$\tau\{t|\mathcal{K}\} = \sum_R S_R(p)\mathcal{H}_R\{p|\mathcal{K}\} \tag{9}$$

and similarly for the torus knots, while for other knots the situation is not that simple. As usual, the concrete solution to the KP equations is specified by the string equation (more generally, by Virasoro/W like constraints). The role of this kind of equation for the knot polynomials is played by the difference equations (A-polynomials) which we discuss in the simplest case of the Jones polynomial in section 4. We end in section 5 with contains some concluding remarks.

2 HOMFLY polynomials as sums of characters

Character expansion of HOMFLY polynomials. Here we outline only the basic idea, details and explanations are given in [6] and forthcoming papers of the series. With a braid representation of the knot we associate the character expansion of the colored HOMFLY polynomial, i.e. represent it as a linear combination of the Schur functions (i.e. the $SU(\infty)/S(\infty)$ characters). Such an expansion depends on the choice of a braid realization, thus, its coefficients by themselves are not knot invariants, instead they are pure group theory quantities and possess many nice properties. For an m-strand braid $\mathcal{B}$ the HOMFLY polynomial in representation R is expanded as

$$H_R^{\mathcal{B}} = \mathrm{tr}\,_{R^{\otimes m}}\Big((q^\rho)^{\otimes m}\mathcal{B}\Big) = \sum_{Q\vdash m|R|} h_R^Q[\mathcal{B}]\, S_Q^*(A) \tag{10}$$

where

$$S_Q^*(A) = \mathrm{tr}\,_{R^{\otimes m}}(q^\rho)^{\otimes m} = S_Q\{p_k^*\}, \tag{11}$$

are the quantum dimensions of representations Q of $SU(N)$. Coefficients $h^Q_R[\mathcal{B}]$ do not depend on A, i.e. on N, thus, they can be evaluated from analysis of arbitrary group $SU_q(N)$.

Instead, these coefficients can be represented as traces in auxiliary spaces of intertwiner operators $\mathcal{M}^Q_{R^m}$, whose dimension is the number $\dim\mathcal{M}^Q_{R^m} = N^Q_{R^m}$ of times the irreducible representation Q appears in the m-th tensor power of the representation R,

$$R^{\otimes m} = \sum_{Q\vdash m|R|} \mathcal{M}^Q_{R^m} \otimes Q \tag{12}$$

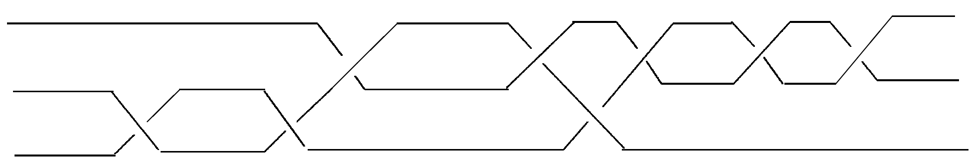
These new traces (which we denote Tr in order to differ from the traces tr in the space of representation) are taken of products of diagonal quantum $\mathcal{R}$-matrices $\widehat{\mathcal{R}}$ acting in $\mathcal{M}^Q_{R^m}$ and the "mixing matrices" intertwining the $\mathcal{R}$-matrices, acting on different pairs of adjacent strands in the braid. These mixing matrices, in their turn, can be represented as products of universal constituents, associated with a switch between two "adjacent" trees, describing various decompositions (12).

HOMFLY for any knot with 2,3,4 braids in the fundamental representation. In [6] we exhaustively described such representations for the coefficients $h^Q_R[\mathcal{B}]$ for *arbitrary* $m = 2,3,4$-strand braids and for the simplest representation $R = [1]$:

$$\begin{gathered}\underline{m=2}, \qquad\qquad \mathcal{B} = \mathcal{R}^a : \\ H^{(a)}_{[1]} = q^a S^*_2(A) + \left(-\frac{1}{q}\right)^a S^*_{11}(A) = q^a S^*_2(A) \ + \ \left(q \longrightarrow -\frac{1}{q}\right)\end{gathered} \tag{13}$$

$$\begin{gathered}\underline{m=3}, \qquad \mathcal{B} = (\mathcal{R}\otimes I)^{a_1}(I\otimes\mathcal{R})^{b_1}(\mathcal{R}\otimes I)^{a_2}(I\otimes\mathcal{R})^{b_2}\ldots : \\ H^{(a_1,b_1,a_2,b_2,\ldots)}_{[1]} = q^{\sum_i(a_i+b_i)} S^*_3(A) + \left(-\frac{1}{q}\right)^{\sum_i(a_i+b_i)} S^*_{111}(A) + \Big(\mathrm{Tr}_{\,2\times2}\widehat{\mathcal{R}}^{a_1}_2 U_2 \widehat{\mathcal{R}}^{b_1}_2 U^\dagger_2 \widehat{\mathcal{R}}^{a_2}_2 U_2 \widehat{\mathcal{R}}^{b_2}_2 U^\dagger_2 \ldots\Big) S^*_{21}(A)\end{gathered} \tag{14}$$

Thus, an arbitrary 3-strand braid is parameterized by a sequence of integers $a_1, b_1, a_2, b_2, \ldots$, their meaning can be understood from the picture (in this figure $a_1 = -2$, $b_1 = 2$, $a_2 = -1$, $b_2 = 3$: this is knot 8_{10}):



Similarly,

$$\underline{m=4}, \qquad \mathcal{B} = (\mathcal{R}\otimes I\otimes I)^{a_1}(I\otimes\mathcal{R}\otimes I)^{b_1}(\mathcal{R}\otimes I\otimes I)^{c_1}(\mathcal{R}\otimes I\otimes I)^{a_2}(I\otimes\mathcal{R}\otimes I)^{b_2}(\mathcal{R}\otimes I\otimes I)^{c_2}\ldots :$$

$$\begin{gathered}H^{(a_1,b_1,c_1,a_2,b_2,c_2,\ldots)}_{[1]} = q^{\sum_i(a_i+b_i+c_i)} S^*_4(A) + \left(-\frac{1}{q}\right)^{\sum_i(a_i+b_i+c_i)} S^*_{1111}(A) + \\ + \Big(\mathrm{Tr}_{\,2\times2}\widehat{\mathcal{R}}^{a_1}_2 U_2 \widehat{\mathcal{R}}^{b_1}_2 U^\dagger_2 \widehat{\mathcal{R}}^{c_1+a_2}_2 U_2 \widehat{\mathcal{R}}^{b_2}_2 U^\dagger_2 \widehat{\mathcal{R}}^{c_2+a_3}_2 \ldots\Big) S^*_{22}(A) + \\ + \left\{\Big(\mathrm{Tr}_{\,3\times3}\widehat{\mathcal{R}}^{a_1}_3 U_3 \widehat{\mathcal{R}}^{b_1}_3 V_3 U_3 \widehat{\mathcal{R}}^{c_1}_3 U^\dagger_3 V^\dagger_3 U^\dagger_3 \widehat{\mathcal{R}}^{a_2}_3 U_3 \widehat{\mathcal{R}}^{b_2}_3 V_3 U_3 \widehat{\mathcal{R}}^{c_2}_3 U^\dagger_3 V^\dagger_3 U^\dagger_3 \ldots\Big) S^*_{31}(A) \ + \ \left(q \longrightarrow -\frac{1}{q}\right)\right\}\end{gathered} \tag{15}$$

In these formulas:

$$\hat{\mathcal{R}}_2 = \begin{pmatrix} q & \\ & -\frac{1}{q} \end{pmatrix} \qquad \hat{\mathcal{R}}_3 = \begin{pmatrix} q & & \\ & q & \\ & & -\frac{1}{q} \end{pmatrix} \tag{16}$$

$$U_2 = \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix} \qquad U_3 = \begin{pmatrix} 1 & & \\ & c_2 & s_2 \\ & -s_2 & c_2 \end{pmatrix} \qquad V_3 = \begin{pmatrix} c_3 & s_3 & \\ -s_3 & c_3 & \\ & & 1 \end{pmatrix} \tag{17}$$

Subscripts refer to the size of the matrices, the entries of rotation matrices U and V are given by

$$c_k = \frac{1}{[k]}, \qquad s_k = \sqrt{1-c_k^2} = \frac{\sqrt{[k-1]\,[k+1]}}{[k]} \tag{18}$$

These formulas provide a very transparent and convenient representation for infinitely many HOMFLY polynomials and seem to be very useful for any theoretical analysis of their general properties, from integrability to linear Virasoro like relations (including A-polynomials, spectral curves, AMM/EO topological recursion etc). They describe in a very effective way the HOMFLY polynomials' dependence on particular a_i, b_i, c_i, i.e. on the shape of the braid. Therefore, further insights are important about the structure of these formulas and their generalizations (in [6] the $m = 5$ case is also investigated, and the general formula for the coefficients $h_{[1]}^{[m-1,1]}$ is suggested for all m).

Colored HOMFLY for torus knots. Especially simple are the HOMFLY polynomials for the torus knots. In this case, the coefficients h_R^Q are known explicitly in far more generality: for all torus knots $[m, n]$ [21]:

$$h_R^Q = q^{\frac{n}{m}\varkappa_Q} C_R^Q \tag{19}$$

where C_R^Q are provided by "the Adams operation":

$$S_R(p^{[m]}) = \sum_Q C_R^Q S_Q(p), \qquad p_k^{[m]} = p_{mk} \tag{20}$$

and

$$\varkappa_Q = \nu_{Q'} - \nu_Q$$
$$\nu_Q = \sum_i (i-1)Q_i, \qquad \varkappa_Q = \frac{1}{2}\sum_i Q_i(Q_i - 2i + 1) = \sum_{(i,j)\in Q} (i-j) \tag{21}$$

(Q' denotes the transposed Young diagram). Similarly, for l-component torus knot the colored HOMFLY polynomials depend on l different representations, and so do the coefficients $C_{R_1\ldots R_l}^Q$ so that the Adams operation reads

$$\prod_{a=1}^{l} S_{R_a}(p^{[m]}) = \sum_Q C_{R_1\ldots R_l}^Q S_Q(p) \tag{22}$$

In fact, for torus knots the β-deformations of eqs.(10),(19),(20) are known [17] which describe the character (MacDonald) decomposition of superpolynomials. It is extremely interesting to find a β-deformation of (13)-(15).

3 Integrability

Continuation from t^* to arbitrary t. One of the main motivations for representation (6) is a possibility to promote the HOMFLY polynomials to KP τ-functions. Namely, for

$$H_R(A) = \sum_Q h_R^Q S_Q^* \tag{23}$$

define

$$\mathcal{H}_R\{t\} = \sum_Q h_R^Q S_Q\{t\} \tag{24}$$

with the same coefficients h_R^Q. Similarly, introduce for a given knot

$$\mathcal{H}\{t|\bar{t}\} = \sum_R \mathcal{H}_R\{t\} S_R(\bar{t}) = \sum_{R,Q} h_R^Q S_R\{\bar{t}\} S_Q\{t\} \tag{25}$$

and for a given link

$$\mathcal{H}\{t|\bar{t}^{(a)}\} = \sum_{R_1\dots R_l} \mathcal{H}_{R_1\dots R_l}\{t\} \prod_{a=1}^{l} S_{R_a}(\bar{t}^{(a)}) = \sum_{R,Q} h^Q_{R_1\dots R_l} \prod_{a=1}^{l} S_{R_a}(\bar{t}^{(a)}) S_Q\{t\} \tag{26}$$

It turns out that this generating function is a KP τ-function of t-variables, in the case of torus knots. Its integrability properties w.r.t. the $\bar{t}$ variables remain to be understood.

Torus knots. In order to study the torus case, let us note that $\varkappa_Q$ is the eigenvalue

$$\hat{W}_{[2]} S_Q(t) = \varkappa_Q S_Q(t) \tag{27}$$

of the simplest cut-and-join operator $\hat{W}_{[2]}$ [22] on the Schur eigenfunction $s_Q\{p\}$ corresponding to the Young diagram Q. It is manifestly given by

$$\hat{W}_{[2]} = \frac{1}{2}\sum_{a,b}\left[(a+b)p_a p_b \frac{\partial}{\partial p_{a+b}} + ab p_{a+b}\frac{\partial^2}{\partial p_a \partial p_b}\right] \tag{28}$$

Then, using the Cauchy formula

$$\sum_R S_R\{t\} S_R\{\bar{t}\} = \exp\sum_k k t_k \bar{t}_k \tag{29}$$

one obtains

$$\mathcal{H}^{[m,n]}\{t,\bar{t}\} = q^{-\frac{n}{m}\hat{W}(t)} e^{\sum_k mkt_{mk}\bar{t}_k} \tag{30}$$

The exponential of t-variables is the simplest KP τ-function. Since the cut-and-join operator $\hat{W}$ is an element of the group $GL(\infty)$, its action preserves KP-integrability in t [10, 23]. Therefore for arbitrary torus knot $[m,n]$ the generating function $\mathcal{H}^{[m,n]}\{t,\bar{t}\}$ is, indeed, the KP τ function in t (but not in $\bar{t}$).

Similarly, the generating function of the torus link, $\mathcal{H}\{t|\bar{t}^{(a)}\}$ is the same τ-function with redefined parameters $\bar{t}_k \to \sum_{a=1}^l \bar{t}_k^{(a)}$:

$$\mathcal{H}^{[m,n]}\{t|\bar{t}^{(a)}\} = q^{-\frac{n}{m}\hat{W}(t)} e^{\sum_k mkt_{mk}\left(\sum_{a=1}^l \bar{t}_k^{(a)}\right)} \tag{31}$$

Non-torus knot/link examples. In the case of non-toric knots the same generating function is typically not a KP τ-function. In order to check this, let us consider the first non-trivial Plücker relation (2) for $g_Q = \sum_R h_R^Q S_R(\bar{t})$ and a 4-strand knot. Then, since $g_0 = 1$, $g_{[1]} = g_{[2]} = g_{[11]} = g_{[21]} = 0$ in this case, in order to satisfy (2), one inevitably should have $g_{[22]} = h_{[1]}^{[22]} = 0$. This is the case for the torus knots, and not typically the case for others. Indeed, for the first 4-strand knots from the Rolfsen table (up to 8 crossings) [5] one has [6]:

knot	$h_{[1]}^{[22]}$
6_1	$q^{-1} - q^1$
7_2	$-q^7 + q^5 - 2q^3 + 3q^1 - 3q^{-1} + 2q^{-3} - q^{-5} + q^{-7}$
7_4	$(q - q^{-1})(q^6 - q^4 + 3q^2 - 1 + 3q^{-2} - q^{-4} + q^{-6})$
7_6	$-q^7 + 2q^5 - 3q^3 + 3q^1 - 3q^{-1} + 3q^{-3} - 2q^{-5} + q^{-7}$
7_7	$-q^7 + 3q^5 - 4q^3 + 5q^1 - 5q^{-1} + 4q^{-3} - 3q^{-5} + q^{-7}$
8_4	$(q - q^{-1})(q^4 - q^2 + 1 - q^{-2} + q^{-4})$
8_6	$(q - q^{-1})(q^2 + 1 + q^{-2})(q^2 - 1 + q^{-2})$
8_{11}	$-q^3 + q^{-3}$
8_{13}	$(q - q^{-1})(q^4 - q^2 + 1 - q^{-2} + q^{-4})$
8_{14}	$(q - q^{-1})(q^2 + 1 + q^{-2})(q^2 - 1 + q^{-2})$
8_{15}	$(q - q^{-1})(q^6 - 2q^4 + 2q^2 - 3 + 2q^{-2} - 2q^{-4} + q^{-6})$

Thus, for all these knots the Plücker relation (2) is not satisfied (torus knots with 4 strands have more than 8 crossings).

4 Difference equations for torus knots in the case of $N = 2$

Knot polynomial as an average. Difference equations, originally nicknamed non-commutative A-polynomials [3] (they are polynomials in powers of the shift operator, changing the heights of the rows in Young diagram R) are examples of *linear* relations between the HOMFLY polynomials H_R associated with different Young diagrams R. They play the same role as "the string equations" in matrix model theory and are presumably a piece of the infinite system of Virasoro like constraints (recursion relations), which still remain to be discovered. They can be used to introduce a spectral curve, then, the AMM/EO topological recursion [7] presumably restores the entire HOMFLY polynomial; by now, this was checked [24] for Jones polynomials in two particular cases of non-torus knots and for the torus knots.

In the previous section, we explained that the character decompositions provide a natural approach for the study of quadratic relations. Now we demonstrate that they are not less useful for the search of linear relations. Again, we restrict our consideration to the torus knots, and also to the case of $SL(2)$ group, i.e. to $A = q^N = q^2$. In this case, non-vanishing are only the Schur polynomials associated with the single-row Young diagrams,

$$S_k[X] = \frac{x_1^{k+1} - x_2^{k+1}}{x_1 - x_2} = x_1^k + x_1^{k-1} x_2 + \ldots + x_2^k \tag{32}$$

and the two-row diagrams, but the latter ones are expressed through the previous ones:

$$S_{[k-l,l]} = S_{k-2l} \tag{33}$$

All other

$$S_{[k-l_1-l_2,l_1,l_2,\dots]} = 0, \qquad \text{for } l_2 \neq 0, \qquad\qquad (34)$$
$$\dots$$

Note, however, that $\varkappa_{[k-l,l]} = (k-l)(k-l-1) + l(l-3) \neq \varkappa_{k-2l} = (k-2l)(k-2l-1)$.

In fact, below we deal with the characters of the simple Lie groups ($SU(2)$ in this case), hence, we slightly rescale the character $S_{[k]} \to \mathrm{S}_{[k]} = S_{[k]}/x_2^k$. This effects just a normalization factor of the knot polynomial.

The difference equation is going to be in the variable k, that is, the height of the single-row Young diagram. The property which we are going to use in the derivation of this equation is that the HOMFLY polynomial H_R for the knot $\mathcal{K}$ represented as an m-strand braid $\mathcal{B}^{\mathcal{K}}$ can be presented as an average over the $N \times N$ matrix $U = e^u$ of the character $\mathrm{S}_R(U^m)$ with some measure which depends on the braid:

$$\mathrm{H}_R^{\mathcal{K}} = \Big\langle \mathrm{S}_R(U^m) \Big\rangle_{\mathcal{B}^{\mathcal{K}}} \qquad (35)$$

Since thus presented HOMFLY polynomial has a specific normalization, we denote it differently.

At least, for the torus knots such a representation does exist, and is explicitly given, for example, by the matrix model [25]. We shall use this concrete model to derive an explicit shape of the difference equation (i.e. of the A-polynomial).

The very fact that an equation exists does not depend on the shape of the measure. Its *raison d'etre* is very simple:

$$\mathrm{H}_{[k+1]}^{\mathcal{K}} - \mathrm{H}_{[k-1]}^{\mathcal{K}} = \Big\langle \mathrm{S}_{[k+1]}(U^m) - \mathrm{S}_{[k-1]}(U^m) \Big\rangle_{\mathcal{B}^{\mathcal{K}}} = \Big\langle e^{m(k+1)(u_1-u_2)} + e^{-m(k+1)(u_1-u_2)} \Big\rangle_{\mathcal{B}^{\mathcal{K}}} = V_k^{\mathcal{B}_{\mathcal{K}}}(q) \qquad (36)$$

where V_k is a $\mathcal{B}_{\mathcal{K}}$-dependent polynomial in q, which can be explicitly evaluated if the measure is known.

$V_k^{[m,n]}$ **from the matrix model.** According to [25], for the torus knot $\mathcal{B}_{\mathcal{K}} = [m,n]$, which is represented as an m-strand braid, the measure is given by

$$\Big\langle \dots \Big\rangle^{[m,n]} = \left(\frac{\eta}{2\pi h}\right)^{N/2} \prod_{i=1}^{N} \int du_i e^{-\frac{\eta u_i^2}{h}} \prod_{i<j} \sinh(u_i - u_j) \sinh\Big(\eta(u_i - u_j)\Big) \Big(\dots\Big) \qquad (37)$$

where $\eta = \frac{m}{n}$ and $q = e^h$.

In the case of $N = 2$ the average of any exponential of u_1 and u_2 is a 4-term polynomial in q, in particular[4],

$$\begin{aligned} V_k^{[m,n]} &= \Big\langle e^{m(k+1)(u_1-u_2)} + e^{-m(k+1)(u_1-u_2)} \Big\rangle_{[m,n]} = \\ &= q^{\frac{m^2+n^2}{2mn}} q^{\frac{mn(k+1)^2}{2}} \left\{ q\Big(q^{(m+n)(k+1)} + q^{-(m+n)(k+1)}\Big) - \frac{1}{q}\Big(q^{(n-m)(k+1)} + q^{(m-n)(k+1)}\Big) \right\} \end{aligned} \qquad (38)$$

This result coincides with the difference equation for the torus knots obtained in [27]. However, this second order difference equation reduces to the first order one for the torus knots of the series $[2, 2s+1]$ [27]. Let us illustrate it in the simplest example of the trefoil.

[4] The character S_k itself is a sum of $k+1$ terms [26]:

$$\Big\langle \mathrm{S}_k(U^m) \Big\rangle_{[m,n]} = \Big\langle \sum_{j=0}^{k} e^{m(k-2j)(u_1-u_2)} \Big\rangle_{[m,n]} =$$

$$= q^{\frac{m^2+n^2}{2mn}} \sum_{j=0}^{k-1} q^{mn(k-2j)^2} \left\{ q\Big(q^{(m+n)(k-2j)} + q^{-(m+n)(k-2j)}\Big) - \frac{1}{q} q\Big(q^{(n-m)(k-2j)} + q^{(m-n)(k-2j)}\Big) \right\}$$

Simplest example of the trefoil $3_1 = [2,3]$. In this case, the Jones polynomial ($l = k+1 = 2j+1$ for the spin j representation) [28]

$$J_l(q) = [l]_q \left(1 + \sum_{i=1}^{l-1} (-)^i q^{-i(i+3)} q^{-2il} \prod_{j=1}^{i} \left(1 - q^{2(l-j)}\right)\left(1 - q^{2(l+j)}\right)\right) \tag{39}$$

satisfies the difference equation [3] (note that we used in similar formulas in [13] different normalization of the Jones polynomials, and also $q \to 1/q$)

$$J_l + q^{-3(2l-1)} J_{l-1} = q^{-3(l-1)} [2l-1]_q \tag{40}$$

It follows that

$$\mathrm{H}_k = q^{3(k+1)^2} q^{1/12-2} \{q\} J_{k+1} \tag{41}$$

satisfies

$$\mathrm{H}_{k+1} + \mathrm{H}_k = q^{3(k^2+3k+3)} [2k+3]_q \{q\} q^{\frac{1}{12}-2} \tag{42}$$

and, taking the difference of two successive equations of this form, one gets

$$\mathrm{H}_{k+1} - \mathrm{H}_{k-1} = q^{\frac{1}{12}+3(k+1)^2+1} \{q\} \left(q^{3(k+1)} [2k+3]_q - q^{-3(k+1)} [2k+1]_q\right) \tag{43}$$

which is exactly $V_k^{[2,3]}$ in (38).

5 Conclusion

This paper is devoted to revealing a significance of the character expansion of the HOMFLY polynomials. We explained that the character decomposition can be defined unambiguously for particular braid representations of the knot. Then, it can be studied in full generality for *all* braids with the particular number m of strands, thus, putting under control the dependence of the HOMFLY polynomial on the shape of the knot.

• We presented explicit results for $m = 2, 3, 4$ from [6], demonstrating the existence of an additional universal hierarchical structure in the formulas.

• It is clear from these examples that the character decomposition provides explicit formulas for the entire *series* of knots depending on arbitrary parameters, thus, opening a possibility to study various hidden relations between knot invariants.

• The character decomposition explicitly separates dependencies on the shape of the knot and the size N of the group. This allows one to continue the formulas from the particular "frozen" values of the hidden time-variables $p_k = p_k^* = \frac{A^k - A^k}{q^k - q^{-k}}$ to arbitrary values of p_k, a trick which already proved extremely useful in the study of the torus *super*polynomials [17].

An open question remains about the further separation of the R-variable (labeling the representation).

These results are immediately applicable to the search of linear and quadratic relations, in particular, of integrability properties of the HOMFLY polynomials and of difference equations ("A-polynomials"), which they satisfy. We explicitly demonstrated such applications in the case of the torus knots and showed that:

• When continued to arbitrary values of p_k, these polynomials become KP τ-functions; this is a non-trivial property, literally correct only for the torus knots.

• Difference equations for the colored Jones polynomials can be derived "in one line" for the arbitrary torus knot $[m, n]$. Moreover, generalizations to HOMFLY in this case are also straightforward.

It would be interesting to find an extension of the matrix model including the time-variables $\{p_k\}$. It should be straightforward, since the W-representation is known and, thus, the methods of [29] can

be applied. One can also pose the question about a complete system of Virasoro like constraints, of which the difference equation should be just the single (lowest?) constituent.

Also of interest is search for a counterpart of (35) for non-torus knots or there can be obstacles for existence of measures with such a property.

The most intriguing is an unambiguous definition of the character decomposition after the β-deformation from HOMFLY to the *super*polynomials. In the case of the torus knots it is recently found in [17], generalization to arbitrary knots is the next point on agenda.

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